STABILITY OF A SPHERICAL CAVITY IN A SOUND FIELD

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The linear theory of the stability of the spherical shape of a cavity and the stability of its radial oscillations in a sound field are discussed. An equation is derived for the amplitudes of the spherical harmonics with allowance for surface tension, viscosity, and compressibility of the surrounding liquid in the Herring–Flynn approximation. The radial pulsation stability is analyzed in the same approximation. The equations derived in the article are subjected to numerical analysis.

The behavior of a cavity in a sound field has been studied in [1-8]. The equation derived from the hydrodynamical equations for the description of the time variation of the radius R of a spherical cavity has been investigated primarily by numerical methods [4-7]. The stability of the behavior of a spherical cavity in a sound field poses an interesting problem. The stability of the spherical shape of a vapor bubble has been investigated in the Noltingk–Neppiras approximation [9, 10]. The stability of the radial pulsations of a gas-filled cavitation bubble has been investigated [11] without regard for the surface tension, viscosity, and compressibility of the liquid. We now carry out a numerical analysis of the stability of a spherical cavity in a sound field.

We consider the stability of the spherical shape of a cavity. Following Plesset [9], we represent the perturbed radius r_s of the sphere in the form

$$r_s = R(t) + a_n(t) S_n$$

where R(t) is the unperturbed radius of the spherical bubble, S_n is the n-th spherical harmonic, and $a_n(t)$ is the amplitude of the n-th harmonic. We assume that $|a_n(t)| \ll R(t)$.

The derivation of the equations describing the variation of the spherical harmonic amplitudes is similar to that in [9]. Using the expression for the velocity potential with regard for the surface tension and viscosity of a compressible liquid [8] in satisfaction of the acoustic-approximation wave equation, we can show that the representation of the perturbed potential by analogy with [9] up to terms of order $1/c_0$ (c_0 is the unperturbed speed of sound in the liquid) is valid. Assuming that the cavity is filled with a gas that obeys the adiabatic equation of state, we obtain the following equation for the unperturbed radius R(t):

$$R\frac{d^2R}{dt^2}\left(1-\frac{2\mu}{\rho_0}-\frac{2}{c_0}\frac{dR}{dt}\right)+\frac{3}{2}\left(\frac{dR}{dt}\right)^2\left(1-\frac{2\mu}{\rho_0}-\frac{4}{3c_0}\frac{dR}{dt}\right)+\frac{1}{\rho_0}[P_\infty-P(R)]-\frac{R}{\rho_0c_0}\frac{dP(R)}{dt}\left(1-\frac{1}{c_0}\frac{dR}{dt}\right)=0 \quad (1)$$

$$P(R)=\left(P_0+\frac{2\sigma}{R_0}\right)\left(\frac{R_0}{R}\right)^{3\gamma}-\frac{2\sigma}{R}-\frac{4\mu}{R}\frac{dR}{dt}$$

$$P_\infty = P_0 - P_m\sin(\omega t)$$

Here P(R) is the pressure on the surface of the bubble, P_∞ is the pressure at infinity, P_0 is the pressure in the unperturbed liquid, P_m and ω are the amplitude and frequency of the external sound field, τ and μ are the coefficients of surface tension and vicosity, γ is the adiabatic exponent, R_0 is the initial radius of the

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cavity, and ρ_0 and ρ_1 are the densities of the liquid and the gas. For the spherical harmonic amplitudes we have

$$\frac{d^2a_n}{dt^2} + B\frac{da_n}{dt} - Aa_n = 0 \tag{2}$$

$$B = \frac{3}{R}\frac{dR}{dt} + n\left[2\mu(n+1)(n+2)/R^2 + \frac{1}{c_0^2}\frac{dP(R)}{dt}\right][n\rho_0+(n+1)\rho_1]^{-1}$$

$$A = \left\{n\rho_0\left[(n-1)\frac{d^2R}{dt^2} - \frac{2(n+1)}{c_0}\frac{dR}{dt}\frac{d^2R}{dt^2} - \frac{2(n+1)}{Rc_0}\left(\frac{dR}{dt}\right)^3 + \right.\right.$$
$$+ \frac{(n+1)}{c_0\rho_0}\frac{dP(R)}{dt} - \frac{R}{c_0^2\rho_0}\frac{d^2P(R)}{dt^2} - \left.\frac{1}{c_0^2\rho_0}\frac{dR}{dt}\frac{dP(R)}{dt}\right] - (n+1)(n+2)\rho_1\frac{d^2R}{dt^2} - \sigma n(n-1)(n+1)(n+2)/R^2 - \tag{3}$$
$$\left. - \frac{4\mu n(n-1)(n+1)}{R^2}\frac{dR}{dt} - \frac{2\mu n(n+1)(n+4)}{Rc_0^2\rho_0}\frac{dP(R)}{dt}\right\} R^{-1}[n\rho_0 + (n+1)\rho_1]^{-1}$$

For $\rho_1 \ll \rho_0$ expression (1) coincides with the Herring–Flynn equation, and for $\mu = 0$ and $c_0 \to \infty$ the expressions for A and B assume the form given in [9].

We now analyze the stability of the spherical shape of the cavity by numerical integration of Eq. (2) for the perturbation harmonic amplitudes and of Eq. (1) for the unperturbed radius R(t). These equations are integrated numerically by the Runge–Kutta method with the integration step chosen automatically to comply with the prescribed computational accuracy. The time-variable step ensured a relative error of 10^{-5}. The correctness of the final results in the case of an incompressible liquid, a constant external field, and a vapor bubble without regard for the viscosity is tested by comparison of the numerical integration data with the analytic solutions given in [10]. We consider bubbles with $R_0 = 10^{-4}$ cm, filled with a gas that obeys an adiabatic law with adiabatic exponent $\gamma = 1.4$. The resonance frequency of the cavity is calculated according to the expression [12, 13]

$$\omega_0 = R_0^{-1}\{3\gamma[P_0 + (1 - 1/3\gamma)\,2\tau/R_0]/\rho_0 - (2\mu/\rho_0R_0)^2\}^{1/2} \tag{4}$$

The compressibility terms have been omitted in (4), because their contribution is negligible in the Herring–Flynn approximation [14].

The following cases are analyzed: $\omega = \omega_0$ (resonance), $\omega = 10\omega_0$, and $\omega = 0.1\omega_0$. The pressure P_0 is considered to be 1 atm. The amplitudes P_m of the external sound field are chosen to defer the collapse of the cavity as long as possible. The liquid density ρ_0 is made equal to 1 g/cm^3, and the density ρ_1 of the gas in the bubble is calculated by means of the Clausius–Clapeyron equation at an initial temperature of 300°K.

The first-derivative term can be eliminated from Eq. (2) for the spherical harmonic amplitudes. The substitution

$$u = a_n \exp\left(\tfrac{1}{2}\int B\,dt\right) \tag{5}$$

reduces Eq. (2) to the form

$$d^2u/dt^2 + I(t)\,u = 0, \qquad I(t) = -A - B/4 - \tfrac{1}{2}\,dB/dt \tag{6}$$

where A and B are evaluated from (3). Representing I(t) by a Fourier cosine series

$$I(t) = \sum_{k=0}^{N} b_k \cos(2\pi kt/T) \tag{7}$$

we obtain the Hill equation from (6). The stability of the solutions of Eq. (6) for $0 \le t \le T$ can be determined (see [15]) from the values of the coefficients b_k in the expansion (7). The values of b_k are calculated in each of the cases treated below and provide an additional tool for our stability analysis.

We now give the results of the numerical analysis of the stability of the spherical shape of a cavity in the Herring–Flynn approximation. Figure 1 illustrates the behavior of $a_n(t)$ in an incompressible liquid for $\omega = 10\omega_0$, $P_m = 100$ atm, and $\tau = \mu = 0$. Curves 1, 2, and 3 correspond to n = 2, 3, and 4. Here the amplitudes of all the harmonics increase in oscillating fashion with the time. The oscillations of the radius R(t) in this case are modulated by the natural frequency of the cavity (Fig. 2, curve 1). The behavior and detailed form of the curves for $a_n(t)$ depend on the relative values of ω and ω_0. It is evident from Fig.

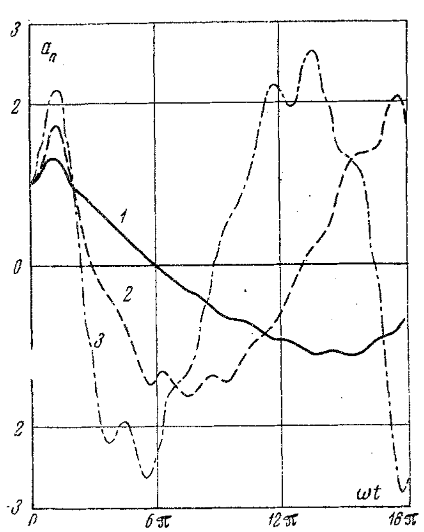

Fig. 1

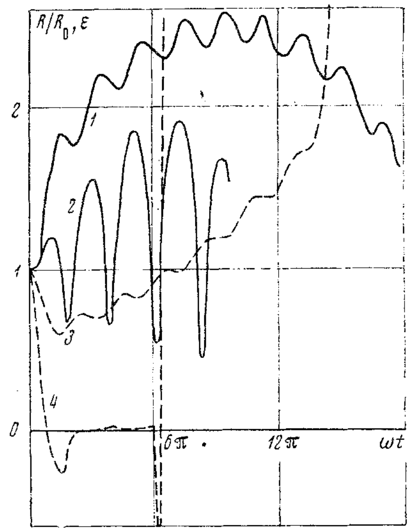

Fig. 2

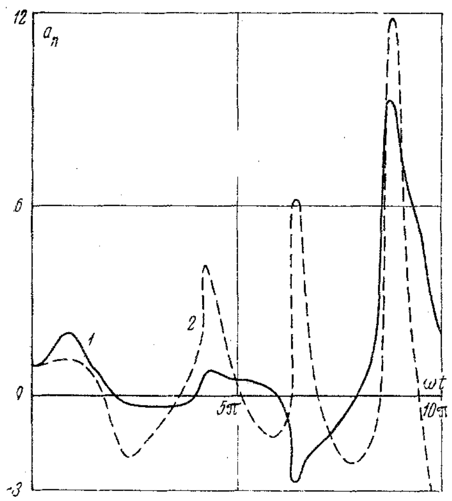

Fig. 3

1 that the profiles of the harmonic amplitudes acquire additional maxima for $\omega = 10\omega_0$. They are absent in the resonance case ($\omega = \omega_0$). This fact is apparent from Fig. 3, which gives curves of $a_n(t)$ in the case of an incompressible liquid for $P_m = 0.5$ atm, $\tau = 0$, and $\mu = 0.002$ cm^2/sec. Curves 1 and 2 correspond to n = 2 and 4. The behavior of R(t) for these parameters is shown in Fig. 2 (curve 2).

When the viscosity is ignored, the amplitudes $a_n(t)$ grow more rapidly with time, the rate of growth increasing with the harmonic order. The surface tension also has a significant effect on the evolution of the perturbation harmonic amplitudes. For $\tau = 75$ dyn/cm, $P_m = 0.5$ atm, and $\mu = 0$ the oscillation amplitudes $a_n(t)$ decrease in the resonance case, and for $\tau = 150$ dyn/cm the amplitudes of the first six harmonics decrease with increasing t. The frequencies of the harmonics increase with the order number n. The inclusion of viscosity, unlike the effect elicited by surface tension, does not cause the frequencies of the harmonics $a_n(t)$ to increase.

With an increase in the pressure P_m, collapse of the bubble takes place in the resonance case. The behavior of the radius in this case has been studied in detail [6]. In an incompressible liquid for $P_m = 300$ atm, $\omega = \omega_0$, and $\tau = \mu = 0$ the amplitudes $a_n(t)$ and frequencies of the harmonics increase as the collapse time is approached. The total amplitudes here are greater than in the case $P_m = 0.5$ atm. For $\tau = 75$ dyn/cm and $\mu = 0$ the qualitative behavior of the spherical harmonic amplitudes remains practically unchanged. An analogous situation occurs for $\tau = 0$ and $\mu = 10^{-2}$ cm^2/sec. In this case the amplitudes $a_n(t)$ decrease for $P_m = 0.5$ atm. The inclusion of compressibility slows the growth of all the harmonic amplitudes. This effect is observed for $\tau = \mu = 0$ and $c_0 = 1.3 \cdot 10^5$ cm/sec. When $P_m = 0.5$ atm, the compressibility increases the growth rates of the second and fourth harmonics and slows the growth of the third, fifth, and sixth harmonics.

For $\omega = 0.1\omega_0$ the spherical shape of the cavity is less stable than in the case $\omega = \omega_0$ or $\omega = 10\omega_0$. Numerical experiments show that the growth rate of $a_n(t)$ is more critically dependent on the harmonic order in unstable cases.

We now consider the stability of radial pulsations of a cavity. We write the following expression for the perturbed bubble radius $\xi(t)$:

$$\xi(t) = R(t) + \varepsilon(t) \tag{8}$$

Here $\varepsilon(t)$ is the perturbation of the radial motion. Assuming $|\varepsilon(t)| \ll R(t)$, substituting $\xi(t)$ into the Herring–Flynn equation (1), and linearizing on $\varepsilon(t)$, we obtain for the perturbation $\varepsilon(t)$

$$\frac{d^2\varepsilon}{dt^2} + \frac{d\varepsilon}{dt}\left\{\frac{3}{R}\frac{dR}{dt} - \frac{1}{\rho_0 c_0}\frac{dP(R)}{dR} + \left[\frac{4\mu}{\rho_0 R^2}\left(1 + \frac{1}{c_0}\frac{dR}{dt}\right) - \frac{2}{c_0}\frac{d^2R}{dt^2}\right] \times \right.$$
$$\left. \times \left(1 - \frac{2}{c_0}\frac{dR}{dt}\right)^{-1}\right\} + \varepsilon\left\{\frac{1}{R}\frac{d^2R}{dt^2} - \left[\frac{1}{\rho_0 R}\frac{dP(R)}{dR}\left(1 + \frac{1}{c_0}\frac{dR}{dt}\right) + \frac{1}{c_0}\frac{dR}{dt}\frac{d^2P(R)}{dR^2}\right]\left(1 - \frac{2}{c_0}\frac{dR}{dt}\right)^{-1}\right\} = 0 \tag{9}$$

We analyze the stability of radial oscillations of the cavity in the Herring–Flynn approximation by numerical integration of the system of equations (1), (9). The values of the parameters (R_0, P_0, etc.) for which the behavior of $\varepsilon(t)$ is considered are indicated above. The amplitude of the sound field is assumed to be 0.5 atm for $\omega = \omega_0$ or $\omega = 0.1\omega_0$ and 100 atm for $\omega = 10\omega_0$.

In the resonance case for $\tau = 0$ and $\mu = 0.002$ cm^2/sec the perturbation $\varepsilon(t)$ decays in the first period and then, beginning with the third period, grows (Fig. 2, curve 4). When surface tension is taken into account, the radial pulsations of the cavity become stable. When $\tau = 150$ dyn/cm, $\varepsilon(t)$ decays in the first period of the sound field. Even greater damping is observed with an increase in the viscosity. The compressibility of the liquid weakly affects the behavior of $\varepsilon(t)$ in the Herring–Flynn approximation. Increasing the amplitude of the external sound field sharpens the instability of the radial pulsations. The inclusion of surface tension and viscosity in this case slows the growth of $\varepsilon(t)$. If the cavity is close to collapse, $\varepsilon(t)$ always increases.

In the case $\omega = 10\omega_0$ for $\tau = \mu = 0$ the perturbation $\varepsilon(t)$ increases with time (Fig. 2, curve 3). If $\tau \neq 0$ and $\mu \neq 0$, the radial motion of the cavity becomes more stable. Here, on the whole, the growth of $\varepsilon(t)$ is faster in the unstable case than for $\omega = \omega_0$.

When $\omega \ll \omega_0$ ($\omega = 0.1\omega_0$), $\varepsilon(t)$ grows during half of the first period. The inclusion of surface tension ($\tau = 75$ dyn/cm) and viscosity ($\mu = 0.01$ cm^2/sec) does not qualitatively change the behavior of $\varepsilon(t)$.

The foregoing numerical analysis of Eqs. (1), (2), and (9) shows that the stability of the radial pulsations and spherical shape of a cavitation bubble depends to varying degrees on the frequency and amplitude of the external sound field, viscosity, compressibility, and surface tension of the liquid. The viscosity and surface tension have a positive effect on the stability, the viscosity effects are more strongly felt at the higher harmonics, and the oscillations of the latter "cut off" with the introduction of dissipation. The surface tension, by contrast with dissipative effects, renders the behavior of the lower harmonics more stable at first. Thus, the viscosity and surface tension make the spherical shape of the cavity more stable, acting on different parts of the perturbation spectrum. The surface tension increases the oscillation frequency of the amplitudes of all the harmonics. The viscosity does not affect the oscillation frequency of the harmonics.

The amplitude of the sound field more strongly affects the stability for $\omega = \omega_0$ and $\omega \ll \omega_0$. Increasing the field amplitude in these cases results in collapse of the cavity. The inclusion of viscosity and surface tension does not remove the instability of the spherical shape of the cavity in this situation.

The radial pulsations and spherical shape of the bubble are the least stable for $\omega \ll \omega_0$. The influence of the viscosity and surface tension is weak in this case.

Allowance for the compressibility of the liquid in the Herring–Flynn approximation does not induce significant changes in the behavior of the spherical harmonic amplitudes and radial pulsations of a cavitation bubble. The compressibility somewhat increases the instability of the spherical shape and radial pulsations except in the event of collapse of the cavity, in which case the compressibility decreases the growth rate of the spherical harmonic amplitudes near the instant of collapse.

LITERATURE CITED

1. Lord Rayleigh, "On pressure developed in a liquid during the collapse of a spherical cavity," Phil. Mag., 34, No. 199 (1917).
2. B. E. Noltingk and E. A. Neppiras, "Cavitation produced by ultrasonics," Proc. Phys. Soc. London, Ser. B, 63, Part 9 (1950).

2a. E. A. Neppiras and B. E. Noltingk, "Cavitation produced by ultrasonics: theoretical conditions for the onset of cavitation," Proc. Phys. Soc. London, Ser. B, 64, Part 12 (1951).

3. R. H. Cole, Underwater Explosions, Princeton Univ. Press (1948).
4. H. G. Flynn, "Physics of acoustic cavitation in liquids," Physical Acoustics (W. P. Mason, ed.), Vol. 1B, Academic Press (1964).
5. M. I. Vorotnikova and R. I. Soloukhin, "A calculation of the pulsations of gas bubbles in an incompressible liquid subject to a periodically varying pressure," Akust. Zh., 10, No. 1, 34 (1964).
6. L. D. Rozenberg (ed.), High-Intensity Ultrasonic Fields, Pt. 2, Plenum (1971).
7. A. D. Pernik, Cavitation Problems [in Russian], Sudpromgiz, Leningrad (1963).
8. F. R. Gilmore, The Growth or Collapse of a Spherical Bubble in a Viscous Compressible Fluid, Heat Transfer and Fluid Mechanics Institute, Stanford Univ. Press (1952).
9. M. S. Plesset, "On the stability of fluid flows with spherical symmetry," J. Appl. Phys., 25, No. 1 (1954).

10. M. S. Plesset and T. P. Mitchell, "On the stability of the spherical shape of a vapor cavity in a liquid," Quart. Appl. Math., 13, No. 4 (1956).
11. A. Eller and H. G. Flynn, "Generation of subharmonics of order one-half by bubbles in a sound field," J. Acoust. Soc. Am., 46, No. 3 (1969).
12. L. Trilling, "The collapse and rebound of a gas bubble," J. Appl. Phys., 23, No. 1 (1952).
13. T. Hirose and M. Okuyama, "Mechanics of the cavitation produced by ultrasonics. II. Conditions for cavitation and stationary cavitation," J. Acoust. Soc. Japan, 13, No. 1 (1957).
14. G. Houghton, "Theory of bubble pulsation and cavitation," J. Acoust. Soc. Am., 35, No. 9 (1963).
15. A. Shima, "The natural frequency of a bubble oscillating in a viscous compressible liquid," Trans. ASME, Ser. D: J. Basic Engng., 92, No. 3, 555 (1970).
16. H. Bateman and A. Erdélyi, Higher Transcendental Functions: Elliptic and Automorphic Functions; Lamé and Mathieu Functions, McGraw-Hill, New York (1953).